Appendix

Proof of Lemma 4.1. Let $1 > \eta > 0$ and $\lambda > 0$, then $\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$ increases with respect to $p > 0$. Further more, if $p = \lambda/(1-\eta)$, then $\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda = \lambda/(1-\eta)$. Therefore if $p \geq \lambda/(1-\eta)$, then $\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda \geq \lambda/(1-\eta) > 0$. On the other hand if $p \geq \lambda/(1-\eta)$, then $p - \left(\eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda\right) = \left(\sqrt{(1-\eta)p} - \sqrt{\lambda}\right)^2 \geq 0$, therefore $p \geq \eta p + 2\sqrt{(1-\eta)p\lambda} - \lambda$. □

Proof of Lemma 4.2. Let $1 > \eta > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $\eta p - 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)\sqrt{p\lambda}$ as $\eta x^2 - 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)ax$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $\eta x^2 - 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)ax = 0$ for x are 0 and $2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)a/\eta$. Therefore if $x > 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)a/\eta$, or equivalently, $x^2 > 8\left(1 - \sqrt{1-\eta^2}\right)a^2/\eta^2$, then $\eta x^2 - 2\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)ax > 0$. Replacing x by $\sqrt{p}$ and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 4.3. Let $1 > \eta > 0$ and $\lambda > 0$, then we have

$$\begin{aligned}
\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)^2 > 0 &\Leftrightarrow 1 - \sqrt{1+\eta}\sqrt{1-\eta} > 0 \\
&\Leftrightarrow 1 + \eta - \sqrt{1+\eta}\sqrt{1-\eta} > \eta \\
&\Leftrightarrow (1+\eta)\left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)^2 > \eta^2 \\
&\Leftrightarrow \left(\sqrt{1+\eta} - \sqrt{1-\eta}\right)^2/\eta^2 > 1/(1+\eta) \\
&\Leftrightarrow 8\left(1 - \sqrt{1-\eta^2}\right)\lambda/\eta^2 > 4\lambda/(1+\eta)
\end{aligned}$$

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Also we have

$$
\begin{aligned}
\left(\sqrt{1+\eta}-\sqrt{1-\eta}\right)^2 > 0 &\Leftrightarrow 0 > \sqrt{1+\eta}\sqrt{1-\eta}-1\\
&\Leftrightarrow \eta > \sqrt{1+\eta}\sqrt{1-\eta}-(1-\eta)\\
&\Leftrightarrow \eta^2 > (1-\eta)\left(\sqrt{1+\eta}-\sqrt{1-\eta}\right)^2\\
&\Leftrightarrow 1/(1-\eta) > \left(\sqrt{1+\eta}-\sqrt{1-\eta}\right)^2/\eta^2\\
&\Leftrightarrow 4\lambda/(1-\eta) > 8\left(1-\sqrt{1-\eta^2}\right)\lambda/\eta^2
\end{aligned}
$$

□

Proof of Lemma 4.4. Let $\eta > 0$ and $\lambda > 0$. If $p > 4\lambda/(1+\eta)$, then $2\sqrt{(1+\eta)p\lambda} - \lambda > 4\lambda - \lambda = 3\lambda > 0$. □

Proof of Lemma 4.5. Let $\eta > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate expression $p - 2\sqrt{(1+\eta)p\lambda} + \lambda$ as $x^2 - 2ax\sqrt{1+\eta} + a^2$ where $x > 0$ and $a > 0$. The solutions to the quadratic equation $x^2 - 2ax\sqrt{1+\eta} + a^2 = 0$ for x are $\left(\sqrt{1+\eta}-\sqrt{\eta}\right)a$ and $\left(\sqrt{1+\eta}+\sqrt{\eta}\right)a$. Therefore if $\left(\sqrt{1+\eta}+\sqrt{\eta}\right)a > x > \left(\sqrt{1+\eta}-\sqrt{\eta}\right)a$, or equivalently, $\left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)a^2 > x^2 > \left(1+2\eta-2\sqrt{\eta(1+\eta)}\right)a^2$, then $0 > x^2 - 2ax\sqrt{1+\eta} + a^2$. Replacing x by $\sqrt{p}$ and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 4.6. Let $\eta > 0$ and $\lambda > 0$. Note that

$$
\begin{aligned}
\sqrt{1+\eta}+\sqrt{\eta} > \sqrt{1+\eta} \Rightarrow 2/\sqrt{1+\eta} > 1/\sqrt{1+\eta} &> 1/\left(\sqrt{1+\eta}+\sqrt{\eta}\right)\\
&\Leftrightarrow 2/(1+\eta) > \sqrt{1+\eta}-\sqrt{\eta}\\
&\Leftrightarrow 4/(1+\eta) > \left(\sqrt{1+\eta}-\sqrt{\eta}\right)^2\\
&\Leftrightarrow 4\lambda/(1+\eta) > \left(1+2\eta-2\sqrt{\eta(1+\eta)}\right)\lambda
\end{aligned}
$$

□

Proof of Lemma 4.7. Let $4/5 > \eta > 0$ and $\lambda > 0$, then we have

$$
\begin{aligned}
4\eta - 5\eta^2 > 0 &\Leftrightarrow 2\sqrt{\eta(1-\eta)} > \eta\\
&\Leftrightarrow 1 + 2\sqrt{\eta(1-\eta)} > 1+\eta\\
&\Leftrightarrow \left(\sqrt{1-\eta}+\sqrt{\eta}\right)^2 > \left(\sqrt{1+\eta}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow \sqrt{1-\eta}+\sqrt{\eta} > \sqrt{1+\eta}\\
&\Leftrightarrow \sqrt{1-\eta} > \sqrt{1+\eta}-\sqrt{\eta}\\
&\Leftrightarrow \sqrt{1-\eta}\left(\sqrt{1+\eta}+\sqrt{\eta}\right) > 1\\
&\Leftrightarrow \left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda > \lambda/(1-\eta)
\end{aligned}
$$

and we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 4.8. Let $4/5 > \eta > 0$ and $\lambda > 0$, then we have

$$
\begin{aligned}
9(1-\eta) > 1+\eta &\Leftrightarrow 3\sqrt{1-\eta} > \sqrt{1+\eta}\\
&\Leftrightarrow 4\sqrt{1-\eta} > \sqrt{1+\eta}+\sqrt{1-\eta}\\
&\Leftrightarrow 2\sqrt{1-\eta}\left(\sqrt{1+\eta}-\sqrt{1-\eta}\right) > \eta\\
&\Leftrightarrow 4(1-\eta)\left(\sqrt{1+\eta}-\sqrt{1-\eta}\right)^2 > \eta^2\\
&\Leftrightarrow 8\left(1-\sqrt{1-\eta^2}\right)\lambda/\eta^2 > \lambda/(1-\eta)
\end{aligned}
$$

and we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 4.13. Let $4/5 > \eta > 0$ and $\lambda > 0$. First we prove (a) and (b) together. According to Lemma 4.8 part (a), $p_3 > p_2$, therefore $p_3 = \eta p_3 + (1-\eta)p_3 > \eta p_3 + (1-\eta)\sqrt{p_2 p_3} > \eta p_2 + (1-\eta)p_2 = p_2$. Next we prove (c). According to Lemma 4.8 part (a), $p_3 > p_2$, therefore $2\eta\left(\sqrt{p_3}-\sqrt{p_2}\right)/\sqrt{p_2} > 0 \Rightarrow \left(1+2\eta\left(\sqrt{p_3}-\sqrt{p_2}\right)/\sqrt{p_2}\right)p_3 > p_3$. Finally we prove (d). According to Lemma 4.3, we have $4p_2 > p_3$, therefore $2\sqrt{p_2} > \sqrt{p_3} \Leftrightarrow \sqrt{p_2} > \sqrt{p_3}-\sqrt{p_2} \Leftrightarrow 1 > \left(\sqrt{p_3}-\sqrt{p_2}\right)/\sqrt{p_2}$. Therefore $(1+2\eta)p_3 > \left(1+2\eta\left(\sqrt{p_3}-\sqrt{p_2}\right)/\sqrt{p_2}\right)p_3$. □

Proof of Lemma 4.14. Let $\eta > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{r}$ and $a \equiv \sqrt{\lambda}$ and the expression $r - 2\sqrt{(1+2\eta)r\lambda/(1+\eta)} + \lambda$ can be restated as $x^2 - 2ax\sqrt{(1+2\eta)/(1+\eta)} + a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $x^2 - 2ax\sqrt{(1+2\eta)/(1+\eta)} + a^2 = 0$ for x are $\left(\sqrt{1+2\eta}-\sqrt{\eta}\right)a/\sqrt{1+\eta}$ and $\left(\sqrt{1+2\eta}+\sqrt{\eta}\right)a/\sqrt{1+\eta}$. Thus if $\left(\sqrt{1+2\eta}+\sqrt{\eta}\right)a/\sqrt{1+\eta} > x > \left(\sqrt{1+2\eta}-\sqrt{\eta}\right)a/\sqrt{1+\eta}$, or equivalently, if $\left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)a/(1+\eta) > x^2 > \left(1+3\eta-2\sqrt{\eta(1+2\eta)}\right)a/(1+\eta)$, then $0 > x^2 - 2ax\sqrt{(1+2\eta)/(1+\eta)} + a^2$. Replacing x by $\sqrt{r}$ and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 4.15. Let $1 > \eta > 0$ and $\lambda > 0$, and we have $\sqrt{1+\eta} > \sqrt{1-\eta}$ and $\sqrt{1+2\eta} > \sqrt{\eta}$, therefore we have $2\sqrt{1+\eta} > \sqrt{1+\eta} +$

$\sqrt{1-\eta}$ and $2\sqrt{1+2\eta} > \sqrt{1+2\eta} + \sqrt{\eta}$. By multiplying these inequalities $4\sqrt{1+2\eta}\sqrt{1+\eta} > \left(\sqrt{1+2\eta}+\sqrt{\eta}\right)\left(\sqrt{1+\eta}+\sqrt{1-\eta}\right)$. By multiplying $\left(\sqrt{1+\eta}-\sqrt{1-\eta}\right)$ and dividing $2\eta\sqrt{1+\eta}$ on both sides we have $2\sqrt{1+2\eta}\left(\sqrt{1+\eta}-\sqrt{1-\eta}\right)/\eta > \left(\sqrt{1+2\eta}+\sqrt{\eta}\right)$. By squaring both sides $(1+2\eta)p_3 > \left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$. □

Proof of Lemma 4.16. Note that $f(x)$ is continuous and differentiable over interval $\left[\sqrt{p_2}, \sqrt{p_3}\right]$:

$$\frac{df(x)}{dx} = -2\eta x - \sqrt{p_2}\left((1-2\eta) - \frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2} = -2\eta - \frac{2r\sqrt{p_2}}{x^3} < 0$$

$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{p_2}} = -\sqrt{p_2} + \frac{r}{\sqrt{p_2}} = \frac{1}{\sqrt{p_2}}(r-p_2)$$

$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{p_3}} = -\sqrt{p_2} - 2\eta(\sqrt{p_3}-\sqrt{p_2}) + \frac{r\sqrt{p_2}}{p_3} = \frac{\sqrt{p_2}}{p_3}(r-r_2)$$

If $r \in (0, p_2]$, then $f(x)$ is decreasing over $\left[\sqrt{p_2}, \sqrt{p_3}\right]$, therefore $x^* = \sqrt{p_2}$. If $r \geq r_2$, then $f(x)$ is increasing over $\left[\sqrt{p_2}, \sqrt{p_3}\right]$, therefore $x^* = \sqrt{p_3}$. If $r \in (p_2, r_2)$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{p_2}$ and decreasing in the neighborhood below $x = \sqrt{p_3}$. Also note that $d^2f(x)/dx^2 < 0$, therefore $x^* \in \left(\sqrt{p_2}, \sqrt{p_3}\right)$ that satisfies the first order condition

$$\left.\frac{df(x)}{dx}\right|_{x=x^*} = 0 \Rightarrow 2\eta\left(x^*\right)^3 + (1-2\eta)\sqrt{p_2}\left(x^*\right)^2 - r\sqrt{p_2} = 0 \tag{A.1}$$

which is a cubic equation. According to the general formula for roots of cubic equation, $x^* = \sqrt{p_{cu}} \in \left(\sqrt{p_2}, \sqrt{p_3}\right)$. □

Proof of Lemma 4.17. Note that $f(x)$ is continuous and differentiable for $x \geq \sqrt{p_3}$:

$$\frac{df(x)}{dx} = -\sqrt{p_1}\left(1+2\eta-\frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2} = -\frac{2r\sqrt{p_1}}{x^3} < 0$$

$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{p_3}} = \frac{\sqrt{p_1}}{p_3}(r-r_3) \text{ and } \lim_{x\to+\infty}\frac{df(x)}{dx} = -(1+2\eta)\sqrt{p_1} < 0$$

If $r \in (0, r_3]$, then $f(x)$ is decreasing for $x \geq \sqrt{p_3}$, therefore $x^* = \sqrt{p_3}$. If $r > r_3$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{p_3}$ and decreasing when x approaches $+\infty$. Also note that $d^2f(x)/dx^2 < 0$, therefore $x^* > \sqrt{p_3}$ that satisfies first order condition $df(x)/dx|_{x=x^*} = 0 \Rightarrow x^* = \sqrt{r/(1+2\eta)}$. □

Proof of Lemma 4.22. Let $\eta > 1/3$ and $\lambda > 0$. Note that

$$\begin{aligned}3\eta > 1 &\Leftrightarrow 2\sqrt{\eta} > \sqrt{1+\eta}\\ &\Leftrightarrow \sqrt{1+\eta} > 2\left(\sqrt{1+\eta} - \sqrt{\eta}\right)\\ &\Leftrightarrow (1+\eta)\left(\sqrt{1+\eta}+\sqrt{\eta}\right)^2 > 4\\ &\Leftrightarrow \left(1+2\eta+2\sqrt{\eta(1+\eta)}\right)\lambda > 4\lambda/(1+\eta)\end{aligned}$$

□

Proof of Lemma 4.25. Note that $f(x)$ is continuous and differentiable for $x \geq \sqrt{p_4}$:

$$\frac{df(x)}{dx} = -\sqrt{p_1}\left(1+2\eta-\frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2} = -\frac{2r\sqrt{p_1}}{x^3} < 0$$
$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{p_4}} = \frac{\sqrt{p_1}}{p_4}(r-r_4) \text{ and } \lim_{x\to+\infty}\frac{df(x)}{dx} = -(1+2\eta)\sqrt{p_1} < 0$$

If $r \in (0, r_4]$, then $f(x)$ is decreasing for $x \geq \sqrt{p_4}$, therefore $x^* = \sqrt{p_4}$. If $r > r_4$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{p_4}$ and decreasing when x approaches $+\infty$. Note that $d^2f(x)/dx^2 < 0$, therefore $x^* > \sqrt{p_4}$ satisfies first order condition $df(x)/dx|_{x=x^*} = 0 \Rightarrow x^* = \sqrt{r/(1+2\eta)}$. □

Proof of Lemma 4.26. Let $\eta > 0$ and $\lambda > 0$, then we have

$$\begin{aligned}&\sqrt{\eta}\left(\sqrt{1+2\eta}-\sqrt{1+\eta}\right)+\eta > 0 \Leftrightarrow 0 > -\sqrt{\eta}\sqrt{1+2\eta}+\sqrt{\eta}\sqrt{1+\eta}-\eta\\ &\Leftrightarrow \sqrt{1+2\eta}\sqrt{1+\eta} > \left(\sqrt{1+2\eta}+\sqrt{\eta}\right)\left(\sqrt{1+\eta}-\sqrt{\eta}\right)\\ &\Leftrightarrow \left(\sqrt{1+\eta}+\sqrt{\eta}\right)\sqrt{1+2\eta} > \left(\sqrt{1+2\eta}+\sqrt{\eta}\right)/\sqrt{1+\eta}\end{aligned}$$

by squaring both sides we have $(1+2\eta)p_4 > \left(1+3\eta+2\sqrt{\eta(1+2\eta)}\right)\lambda/(1+\eta)$.□

Proof of Lemma 5.2. Let $1 > \overline{\eta} > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $\overline{\eta}p/2 + 2\sqrt{(1-\overline{\eta})p\lambda} - \lambda$ as $\overline{\eta}x^2/2 + 2ax\sqrt{(1-\overline{\eta})} - a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $\overline{\eta}x^2/2 + 2ax\sqrt{(1-\overline{\eta})} - a^2 = 0$ for x are $0 > -2\left(\sqrt{1-\overline{\eta}/2}+\sqrt{1-\overline{\eta}}\right)a/\overline{\eta}$ and $2\left(\sqrt{1-\overline{\eta}/2}-\sqrt{1-\overline{\eta}}\right)a/\overline{\eta} > 0$. Therefore if $2\left(\sqrt{1-\overline{\eta}/2}-\sqrt{1-\overline{\eta}}\right)a/\overline{\eta} > x > 0$, or equivalently, $4\left(\sqrt{1-\overline{\eta}/2}-\sqrt{1-\overline{\eta}}\right)^2 a^2/\overline{\eta}^2 > x^2 > 0$, then $0 >$

$\overline{\eta}x^2/2 + 2ax\sqrt{(1-\overline{\eta})} - a^2$. Replacing x by $\sqrt{p}$ and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 5.3. Let $1 > \overline{\eta} > 0$ and $\lambda > 0$, then we have

$$\begin{aligned} 0 > \overline{\eta}^2 - 2\overline{\eta} &\Leftrightarrow \overline{\eta}^2 - 4\overline{\eta} + 4 > 2\overline{\eta}^2 - 6\overline{\eta} + 4 \\ &\Leftrightarrow (2-\overline{\eta})^2 > 4(1-\overline{\eta}/2)(1-\overline{\eta}) \\ &\Leftrightarrow 2-\overline{\eta} > 2\sqrt{1-\overline{\eta}/2}\sqrt{1-\overline{\eta}} \\ &\Leftrightarrow \overline{\eta} > 2\sqrt{1-\overline{\eta}/2}\sqrt{1-\overline{\eta}} - 2(1-\overline{\eta}) \\ &\Leftrightarrow 1/\sqrt{1-\overline{\eta}} > 2\left(\sqrt{1-\overline{\eta}/2} - \sqrt{1-\overline{\eta}}\right)/\overline{\eta} \\ &\Leftrightarrow \lambda/(1-\overline{\eta}) > 4\left(\sqrt{1-\overline{\eta}/2} - \sqrt{1-\overline{\eta}}\right)^2 \lambda/\overline{\eta}^2 \end{aligned}$$

□

Proof of Lemma 5.4. Let $2 > \overline{\eta} > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $(1-\overline{\eta}/2)p - 2\sqrt{p\lambda} + \lambda$ as $(1-\overline{\eta}/2)x^2 - 2ax + a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $(1-\overline{\eta}/2)x^2 - 2ax + a^2 = 0$ for x are $\sqrt{2}a/\left(\sqrt{2}+\sqrt{\overline{\eta}}\right) > 0$ and $\sqrt{2}a/\left(\sqrt{2}-\sqrt{\overline{\eta}}\right) > 0$. Therefore if $\sqrt{2}a/\left(\sqrt{2}-\sqrt{\overline{\eta}}\right) > x > \sqrt{2}a/\left(\sqrt{2}+\sqrt{\overline{\eta}}\right)$, or equivalently, $2a^2/\left(2+\overline{\eta}-2\sqrt{2\overline{\eta}}\right) > x^2 > 2a^2/\left(2+\overline{\eta}+2\sqrt{2\overline{\eta}}\right)$, then $0 > (1-\overline{\eta}/2)x^2 - 2ax + a^2$. Replacing x by $\sqrt{p}$ and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 5.5. Let $\overline{\eta} > 0$ and $\lambda > 0$, then $1 + \sqrt{2\overline{\eta}} > 0 \Leftrightarrow 2 + \sqrt{2\overline{\eta}} > 1 \Leftrightarrow \sqrt{2} > 1/\left(\sqrt{2}+\sqrt{\overline{\eta}}\right) \Leftrightarrow 4\lambda > 2\lambda/\left(2+\overline{\eta}+2\sqrt{2\overline{\eta}}\right)$. □

Proof of Lemma 5.6. Let $8/9 > \overline{\eta} > 0$ and $\lambda > 0$, then we have

$$\begin{aligned} 8\overline{\eta} - 9\overline{\eta}^2 > 0 &\Leftrightarrow 2\sqrt{2\overline{\eta}} > 3\overline{\eta} \\ &\Leftrightarrow 2 - 2\overline{\eta} > 2 + \overline{\eta} - 2\sqrt{2\overline{\eta}} \\ &\Leftrightarrow \sqrt{2}\sqrt{1-\overline{\eta}} > \sqrt{2} - \sqrt{\overline{\eta}} \\ &\Leftrightarrow \sqrt{2}/\left(\sqrt{2}-\sqrt{\overline{\eta}}\right) > 1/\sqrt{1-\overline{\eta}} \\ &\Leftrightarrow 2\lambda/\left(2+\overline{\eta}-2\sqrt{2\overline{\eta}}\right) > \lambda/(1-\overline{\eta}) \end{aligned}$$

and we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 5.7. Let $1 > \overline{\eta} > 0$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate expression $\overline{\eta}p/2 - 2\left(1-\sqrt{1-\overline{\eta}}\right)\sqrt{p\lambda}$ as $\overline{\eta}x^2/2 - 2\left(1-\sqrt{1-\overline{\eta}}\right)ax$ with $x > 0$ and $a > 0$. The solution to the quadratic equation $\overline{\eta}x^2/2 - 2\left(1-\sqrt{1-\overline{\eta}}\right)ax = 0$ for x are 0 and $4\left(1-\sqrt{1-\overline{\eta}}\right)a/\overline{\eta} > 0$. Therefore if $4\left(1-\sqrt{1-\overline{\eta}}\right)a/\overline{\eta} > x > 0$, or equivalently, $16\left(2-\overline{\eta}-2\sqrt{1-\overline{\eta}}\right)a^2/\overline{\eta}^2 > x^2 > 0$, then $0 > \overline{\eta}x^2/2 - 2\left(1-\sqrt{1-\overline{\eta}}\right)ax$. Replacing x by $\sqrt{p}$ and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 5.8. Let $8/9 > \overline{\eta} > 0$ and $\lambda > 0$, then we have

$$\begin{aligned}
0 > 9\overline{\eta}^2 - 8\overline{\eta} &\Leftrightarrow 16(1-\overline{\eta}) > 9\overline{\eta}^2 - 24\overline{\eta} + 16 \\
&\Leftrightarrow 4\sqrt{1-\overline{\eta}} > 4 - 3\overline{\eta} \\
&\Leftrightarrow 4\left(1-\sqrt{1-\overline{\eta}}\right)/\overline{\eta} > 1/\sqrt{1-\overline{\eta}} \\
&\Leftrightarrow 16\left(2-\overline{\eta}-2\sqrt{1-\overline{\eta}}\right)^2 \lambda/\overline{\eta}^2 > \lambda/(1-\overline{\eta})
\end{aligned}$$

and we obtain (a). The proofs for (b) and (c) are similar. □

Proof of Lemma 5.9. Let $1 > \overline{\eta} > 0$ and $\lambda > 0$, then $1 > \sqrt{1-\overline{\eta}} \Leftrightarrow \overline{\eta} > 2\sqrt{1-\overline{\eta}} - 2 + 2\overline{\eta} \Leftrightarrow 4\lambda/(1-\overline{\eta}) > 16\left(2-\overline{\eta}-2\sqrt{1-\overline{\eta}}\right)\lambda/\overline{\eta}^2$. Also note that $1 > \sqrt{1-\overline{\eta}} \Leftrightarrow 2-\overline{\eta}-2\sqrt{1-\overline{\eta}} > 0 \Leftrightarrow 2\left(1-\sqrt{1-\overline{\eta}}\right) > \overline{\eta} \Leftrightarrow 16\left(2-\overline{\eta}-2\sqrt{1-\overline{\eta}}\right)\lambda/\overline{\eta}^2 > 4\lambda$. □

Proof of Lemma 5.14. Let $8/9 > \overline{\eta} > 0$ and $\lambda > 0$. According to Lemma 5.8 and 5.9, $4\overline{p}_1 > \overline{p}_2 > \overline{p}_1$, therefore:

$$\begin{aligned}
2\sqrt{\overline{p}_1} > \sqrt{\overline{p}_2} &\Leftrightarrow \sqrt{\overline{p}_1} > \sqrt{\overline{p}_2} - \sqrt{\overline{p}_1} > 0 \\
&\Leftrightarrow 1 > \left(\sqrt{\overline{p}_2} - \sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1} \\
&\Leftrightarrow \overline{\eta} > \overline{\eta}\left(\sqrt{\overline{p}_2} - \sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1} \\
&\Leftrightarrow 1 > (1-\overline{\eta}) + \overline{\eta}\left(\sqrt{\overline{p}_2} - \sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1} \\
&\Leftrightarrow \overline{p}_2 > (1-\overline{\eta})\overline{p}_2 + \overline{\eta}\overline{p}_2\left(\sqrt{\overline{p}_2} - \sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1}
\end{aligned}$$

Since $\overline{\eta}\overline{p}_2 + (1-\overline{\eta})\sqrt{\overline{p}_1\overline{p}_2} - \overline{\eta}\overline{p}_2\sqrt{\overline{p}_2}/2\left(\sqrt{\overline{p}_2} - \sqrt{\overline{p}_1}\right) = (1-\overline{\eta})\sqrt{\overline{p}_1\overline{p}_2} + \overline{\eta}\overline{p}_2\left(\sqrt{\overline{p}_2} - \sqrt{4\overline{p}_1}\right)/2\left(\sqrt{\overline{p}_2} - \sqrt{\overline{p}_1}\right)$. Since $4\overline{p}_1 > \overline{p}_2 > \overline{p}_1$, therefore we have $(1-\overline{\eta})\overline{p}_2 > (1-\overline{\eta})\sqrt{\overline{p}_1\overline{p}_2}$ and $\overline{\eta}\overline{p}_2\left(\sqrt{\overline{p}_2} - \sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1} > 0 >$

$\overline{\eta}\overline{p}_2\left(\sqrt{\overline{p}_2}-\sqrt{4\overline{p}_1}\right)/2\left(\sqrt{\overline{p}_2}-\sqrt{\overline{p}_1}\right)$. Thus $(1-\overline{\eta})\overline{p}_2+\overline{\eta}\overline{p}_2\left(\sqrt{\overline{p}_2}-\sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1}>\overline{\eta}\overline{p}_2+(1-\overline{\eta})\sqrt{\overline{p}_1\overline{p}_2}-\overline{\eta}\overline{p}_2\sqrt{\overline{p}_2}/2\left(\sqrt{\overline{p}_2}-\sqrt{\overline{p}_1}\right)$. Therefore we obtain (a). According to Lemma 5.8, $\overline{p}_2>\overline{p}_1$, therefore $(1-\overline{\eta})(\overline{p}_2-\overline{p}_1)+\overline{\eta}\overline{p}_2\left(\sqrt{\overline{p}_2}-\sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1}>0 \Leftrightarrow (1-\overline{\eta})\overline{p}_2+\overline{\eta}\overline{p}_2\left(\sqrt{\overline{p}_2}-\sqrt{\overline{p}_1}\right)/\sqrt{\overline{p}_1}>(1-\overline{\eta})\overline{p}_1=\lambda$. Therefore we obtain (b). □

Proof of Lemma 5.15. Note that $f(x)$ is continuous and differentiable over interval $\left[\sqrt{\overline{p}_1},\sqrt{\overline{p}_2}\right]$:

$$\frac{df(x)}{dx}=-\overline{\eta}x-\sqrt{\overline{p}_1}\left((1-2\overline{\eta})-\frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2}=-\overline{\eta}-\frac{2r\sqrt{\overline{p}_1}}{x^3}<0$$

$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{\overline{p}_1}}=-(1-\overline{\eta})\sqrt{\overline{p}_1}+\frac{r}{\sqrt{\overline{p}_1}}=\frac{1}{\sqrt{\overline{p}_1}}(r-\lambda)$$

$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{\overline{p}_2}}=-\overline{\eta}\sqrt{\overline{p}_2}-(1-2\overline{\eta})\sqrt{\overline{p}_1}+\frac{r\sqrt{\overline{p}_1}}{\overline{p}_2}=\frac{\sqrt{\overline{p}_1}}{\overline{p}_2}(r-\overline{r}_2)$$

If $r\in(0,\lambda]$, then $f(x)$ is decreasing over $\left[\sqrt{\overline{p}_1},\sqrt{\overline{p}_2}\right]$, therefore $x^*=\sqrt{\overline{p}_1}$. If $r\geq\overline{r}_2$, then $f(x)$ is increasing over $\left[\sqrt{\overline{p}_1},\sqrt{\overline{p}_2}\right]$, therefore $x^*=\sqrt{\overline{p}_2}$. If $r\in(\lambda,\overline{r}_2)$, then $f(x)$ is increasing in the neighborhood above $x=\sqrt{\overline{p}_1}$ and is decreasing in the neighborhood below $x=\sqrt{\overline{p}_2}$. Also note that $d^2f(x)/dx^2<0$, therefore $x^*\in\left(\sqrt{\overline{p}_1},\sqrt{\overline{p}_2}\right)$ that satisfies the first order condition

$$\left.\frac{df(x)}{dx}\right|_{x=x^*}=0\Rightarrow\overline{\eta}\left(x^*\right)^3+(1-2\overline{\eta})\sqrt{\overline{p}_1}\left(x^*\right)^2-r\sqrt{\overline{p}_1}=0 \tag{A.2}$$

which is a cubic equation. According to the general formula for roots of cubic equation, $x^*=\sqrt{\overline{p}_{cu}}\in\left(\sqrt{\overline{p}_1},\sqrt{\overline{p}_2}\right)$. □

Proof of Lemma 5.16. Note that $f(x)$ is continuous and differentiable for $x\geq\sqrt{\overline{p}_2}$:

$$\frac{df(x)}{dx}=-\sqrt{\lambda}\left(1-\frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2}=-\frac{2r\sqrt{\lambda}}{x^3}<0$$

$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{\overline{p}_2}}=\frac{\sqrt{\lambda}}{\overline{p}_2}(r-\overline{p}_2) \text{ and } \left.\frac{df(x)}{dx}\right|_{x\to+\infty}=-\sqrt{\lambda}<0$$

If $r\in(0,\overline{p}_2]$, then $f(x)$ is decreasing for $r\geq\sqrt{\overline{p}_2}$, therefore $x^*=\sqrt{\overline{p}_2}$. If $r>\sqrt{\overline{p}_2}$, then $f(x)$ is increasing in the neighborhood above $x=\sqrt{\overline{p}_2}$. Since $f(x)$ is decreasing as $x\to+\infty$ and since $f(x)$ is concave ($d^2f(x)/dx^2<0$), therefore $x^*>\sqrt{\overline{p}_2}$ is solved from first order condition $df(x)/dx|_{x=x^*}=0\Rightarrow x^*=\sqrt{r}$. □

Proof of Lemma 5.18. Let $2 > \overline{\eta} \geq 8/9$ and $\lambda > 0$, then we have

$$\begin{aligned} 2\overline{\eta} \geq 16/9 &\Leftrightarrow \sqrt{2\overline{\eta}} \geq 4/3 > 1 \\ &\Leftrightarrow 1 > 2 - \sqrt{2\overline{\eta}} \\ &\Leftrightarrow \sqrt{2}/\left(\sqrt{2} - \sqrt{\overline{\eta}}\right) > 2 \\ &\Leftrightarrow 2\lambda/\left(2 + \overline{\eta} - 2\sqrt{2\overline{\eta}}\right) > 4\lambda \end{aligned}$$

□

Proof of Lemma 5.21. Note that $f(x)$ is continuous and differentiable for $x \geq \sqrt{\overline{p}_3}$:

$$\frac{df(x)}{dx} = -\sqrt{\lambda}\left(1 - \frac{r}{x^2}\right) \text{ and } \frac{d^2f(x)}{dx^2} = -\frac{2r\sqrt{\lambda}}{x^3} < 0$$

$$\left.\frac{df(x)}{dx}\right|_{x=\sqrt{\overline{p}_3}} = \frac{\sqrt{\lambda}}{\overline{p}_3}(r - \overline{p}_3) \text{ and } \left.\frac{df(x)}{dx}\right|_{x\to+\infty} = -\sqrt{\lambda} < 0$$

If $r \in (0, \overline{p}_3]$, then $f(x)$ is decreasing for $x \geq \sqrt{\overline{p}_3}$, therefore $x^* = \sqrt{\overline{p}_3}$. If $r > \sqrt{\overline{p}_3}$, then $f(x)$ is increasing in the neighborhood above $x = \sqrt{\overline{p}_3}$. Since $f(x)$ is decreasing as $x \to +\infty$ and since $f(x)$ is concave ($d^2f(x)/dx^2 < 0$), therefore $x^* > \sqrt{\overline{p}_3}$ is solved from first order condition $df(x)/dx|_{x=x^*} = 0 \Rightarrow x^* = \sqrt{r}$. □

Proof of Lemma 5.23. Let $\overline{\eta} > 2$ and $\lambda > 0$. Define $x \equiv \sqrt{p}$ and $a \equiv \sqrt{\lambda}$ and restate the expression $(1 - \overline{\eta}/2)p - 2\sqrt{p\lambda} + \lambda$ as $(1 - \overline{\eta}/2)x^2 - 2ax + a^2$ with $x > 0$ and $a > 0$. The solutions to the quadratic equation $(1 - \overline{\eta}/2)x^2 - 2ax + a^2 = 0$ for x are $\sqrt{2}a/\left(\sqrt{2} + \sqrt{\overline{\eta}}\right) > 0$ and $0 > \sqrt{2}a/\left(\sqrt{2} - \sqrt{\overline{\eta}}\right)$. Therefore if $\sqrt{2}a/\left(\sqrt{2} + \sqrt{\overline{\eta}}\right) > x > 0$, or equivalently, $2a^2/\left(2 + \overline{\eta} + 2\sqrt{2\overline{\eta}}\right) > x^2 > 0$, then $(1 - \overline{\eta}/2)x^2 - 2ax + a^2 > 0$. Replacing x by $\sqrt{p}$ and a by $\sqrt{\lambda}$ we obtain (a). The proofs for (b) and (c) are similar. □